23/7/2021

Ques Find the real root of the equation $x^3 - 9x + 1 = 0$ by Regula-Falsi Method.

Soln Let $f(x) = x^3 - 9x + 1 = 0$

$f(1) = 1 - 9 + 1 = -7 < 0$

$f(2) = 8 - 9(2) + 1 = -9 < 0$

$f(3) = 3^3 - 9(3) + 1 = 1 > 0$

Since f(2) and f(3) are of opposite signs

∴ the root lies b/w 2 and 3.

So taking $x_0 = 2$, $f(x_0) = -9$

$x_1 = 3$, $f(x_1) = 1$

By Regula Falsi Method we have

$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} (f(x_0))$$

$$= 2 - \frac{(3-2) \times (-9)}{1+9}$$

$$= 2.9$$

$f(x_2) = (2.9)^3 - 9(2.9) + 1$

$= -0.771 < 0$

Thus the root lies b/w 2.9 and 3

So taking $x_0 = 2.9$, $x_1 = 3$, $f(x_0) = -0.771$

$f(x_1) = 1$

Then $$x_3 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} (f(x_0))$$

$$= 2.9 - \frac{(3 - 2.9)}{1 + 0.771} (-0.771)$$

$$= 2.9416$$

$f(x_3) = (2.9416)^3 - 9(2.9416) + 1$

$= -0.0207 < 0$

Thus the root lies b/w 2.9416 & 3

$\therefore$ taking $x_0 = 2.9416$, $x_1 = 3$, $f(x_0) = -0.0207$

$f(x_1) = 1$

$$x_4 = (2.9416)^3 - \frac{(3 - 2.9416)}{1 + 0.0207}(-0.0207)$$

$$= 2.9428$$

$$f(x_4) = (2.9428)^3 - 9(2.9428) + 1$$

$$= -0.0003 < 0$$

Thus the root lies b/w 2.9428 & 3

taking $x_0 = 2.9428$, $x_1 = 3$, $f(x_0) = -0.0003$, $f(x_1) = 1$

$$x_5 = 2.9428 - \frac{(3 - 2.9428)}{1 + 0.0003}(-0.0003)$$

$$= 2.942817$$

Hence the root corrected to four decimal places for given eqn is 2.9428.

Ans

Ques Find the real root of the equation $x^2 - 2x - 1 = 0$ lies b/w 1 and 3 correct upto three decimal places.

Soln

$$f(x) = x^2 - 2x - 1$$

$$f(1) = 1^2 - 2(1) - 1 = -2$$

$$f(3) = 3^2 - 2(3) - 1 = 2.$$

$x_0 = 1$, $f(x_0) = -2$, $x_1 = 3$, $f(x_1) = 2$

Now By Regula Falsi method we have

$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0)$$

$$= 1 - \frac{(3-1)}{2+2}(-2) = 1 + \frac{4}{4} = 2$$

$\therefore f(x_2) = 2^2 - 2(2) - 1$

$= -1 < 0$

$\therefore$ The root lies b/w 2 and 3

$x_0 = 2, x_1 = 3, f(x_0) = -1, f(x_1) = 2$

$x_3 = 2 - \frac{(3-2)}{2+1}(-1)$

$= 2.3333$

$f(x_3) = (2.3333)^2 - 2(2.3333) - 1$

$= -0.2223 < 0$

$\therefore$ The root lies b/w 2.3333 & 3

$x_0 = 2.3333, x_1 = 3, f(x_0) = -0.2223$

$f(x_1) = 2.$

$x_4 = 2.3333 - \frac{(3 - 2.3333)(-0.2223)}{2+0.2223}$

$= 2.3333 + 0.0667$

$= 2.4$

$f(x_4) = (2.4)^2 - 2(2.4) - 1$

$= 5.76 - 4.8 - 1$

$= -0.04 < 0$

Taking $x_0 = 2.4, x_1 = 3, f(x_0) = -0.04$

$f(x_1) = 2$

$x_5 = 2.4 - \frac{(3-2.4)}{(1+0.04)}(-0.04)$

$= 2.4 - (((3-2.4)/(1+0.04))(-0.04))$

$= 2.4 + 0.0118 = 2.4118$

$f(x_5) = (2.4118)^2 - 2(2.4118) - 1$

$= -0.0068$

Taking $x_0 = 2.4118$, $x_1 = 3$, $f(x_0) = -0.0068$

$f(x_1) = 2$.

$$x_6 = 2.4118 + \frac{(3-2.4118)}{2+0.0068}(0.0068)$$

$= 2.4118 + 0.0019$

$= 2.4137 \approx 2.414$

$f(x_6) = (2.4137)^2 - 2(2.4137) - 1$

$= -0.0015$

Taking $x_0 = 2.4137$, $x_1 = 3$, $f(x_0) = -0.0015$

$f(x_1) = 2$.

$$x_7 = 2.4137 + \frac{(3-2.4137)}{2+0.0015}(0.0015)$$

$= 2.4137 + 0.00043$

$= 2.4141 \approx 2.414$

~~Taking $x_0 = 2.4141$, $x_1 = 3$,~~

$f(x_7) = (2.4141)^2 - 2(2.4141) - 1$

$= -0.003$

Taking $x_0 = 2.4141$, $x_1 = 3$, $f(x_0) = (-0.003)$, $f(x_1) = 2$

$$x_8 = 2.4141 + \frac{3-2.4141}{1+0.003}(0.003)$$

$\simeq 2.4158$

The root of given eqn. has an approximate value upto three decimal places of 2.414

3.5.1. Rate of Convergence of Secant Method :

By Regula-Falsi method, we have

$$x_{n+1} = x_n - \frac{f(x_n)\,[x_n - x_{n-1}]}{f(x_n) - f(x_{n-1})}. \qquad \text{...(1)}$$

Let ξ be the exact root of the equation $f(x) = 0$ and let ε_{n-1}, ε_n and ε_{n+1} be the errors corresponding to the approximation x_{n-1}, x_n and x_{n+1} respectively. Then

$$f(\xi) = 0 \qquad \text{...(2)}$$

$$\left.\begin{aligned} x_{n-1} &= \xi + \varepsilon_{n-1} \\ x_n &= \xi + \varepsilon_n \\ x_{n+1} &= \xi + \varepsilon_{n+1} \end{aligned}\right\}. \qquad \text{...(3)}$$

Substituting the values of x_{n-1}, x_n and x_{n+1} from (3) into (1), we get

$$\xi + \varepsilon_{n+1} = \xi + \varepsilon_n - \frac{f(\xi + \varepsilon_n)\,[\xi + \varepsilon_n - \xi - \varepsilon_{n-1}]}{f(\xi + \varepsilon_n) - f(\xi + \varepsilon_{n-1})}$$

or

$$\varepsilon_{n+1} = \varepsilon_n - \frac{(\varepsilon_n - \varepsilon_{n-1})\,f(\xi + \varepsilon_n)}{f(\xi + \varepsilon_n) - f(\xi + \varepsilon_{n-1})}. \qquad \text{...(4)}$$

Equation (4) may be rewritten as

$$\varepsilon_{n+1} = \frac{\varepsilon_{n-1}\,f(\xi + \varepsilon_n) - \varepsilon_n\,f(\xi + \varepsilon_{n-1})}{f(\xi + \varepsilon_n) - f(\xi + \varepsilon_{n-1})}. \qquad \text{...(5)}$$

Now expand $f(\xi + \varepsilon_n)$ and $f(\xi + \varepsilon_{n-1})$ by Taylor's series, we get

$$f(\xi + \varepsilon_n) = f(\xi) + \varepsilon_n f'(\xi) + \frac{\varepsilon_n^{\,2}}{2\,!} f''(\xi) + \ldots$$

$$= \varepsilon_n f'(\xi) + \frac{\varepsilon_n^{\,2}}{2} f''(\xi) + \ldots \qquad [\because\ f(\xi) = 0]$$

Similarly, $f(\xi + \varepsilon_{n-1}) = \varepsilon_{n-1} f'(\xi) + \frac{\varepsilon_{n-1}^2}{2} f''(\xi) + \ldots$.

Putting the values of $f(\xi + \varepsilon_n)$ and $f(\xi + \varepsilon_{n-1})$ in (5), we get

$$\varepsilon_{n+1} = \frac{\varepsilon_{n-1}\left[\varepsilon_n f'(\xi) + \frac{\varepsilon_n^{\,2}}{2} f''(\xi) + \ldots\right] - \varepsilon_n\left[\varepsilon_{n-1} f'(\xi) + \frac{\varepsilon_{n-1}^2}{2} f''(\xi) + \ldots\right]}{\left[\varepsilon_n f'(\xi) + \frac{\varepsilon_n^{\,2}}{2} f''(\xi) + \ldots\right] - \left[\varepsilon_{n-1} f'(\xi) + \frac{\varepsilon_{n-1}^2}{2} f''(\xi) + \ldots\right]}$$

$$= \frac{\frac{1}{2}\varepsilon_{n-1}\varepsilon_n\,(\varepsilon_n - \varepsilon_{n-1})\,f''(\xi) + \ldots}{(\varepsilon_n - \varepsilon_{n-1})\,f'(\xi) + \frac{1}{2}(\varepsilon_n^{\,2} - \varepsilon_{n-1}^2)\,f''(\xi) + \ldots}$$

$$\varepsilon_{n+1} = \frac{\frac{1}{2}\varepsilon_{n-1}\,\varepsilon_n\,(\varepsilon_n - \varepsilon_{n-1})\,f''(\xi)}{(\varepsilon_n - \varepsilon_{n-1})\,f'(\xi)}$$

[Ignoring ε_{n-1}^2 and $\varepsilon_n^{\,2}$ and their higher power terms] or $$\varepsilon_{n+1} = \frac{1}{2}\varepsilon_{n-1}\varepsilon_n \frac{f''(\xi)}{f'(\xi)}$$

or $$\varepsilon_{n+1} = \varepsilon_{n-1}\varepsilon_n k, \text{ where } k = \frac{f''(\xi)}{f'(\xi)}. \quad ...(6)$$

Let m be the rate of convergence of the method. Then

$$\varepsilon_n = \varepsilon_{n-1}^m k_1, \text{ where } k_1 \text{ is a constant} \quad ...(7)$$

Eliminating ε_{n-1} between (6) and (7), we get

$$\varepsilon_{n+1}^m = \frac{\varepsilon_n^{m+1} k^m}{k_1}$$

or $$\varepsilon_{n+1} = \varepsilon_n^{(1+1/m)} \frac{k}{(k_1)^{1/m}}. \quad ...(8)$$

From (7), $$\varepsilon_{n+1} = \varepsilon_n^m k_1. \quad ...(9)$$

Now eliminating ε_{n+1} between (8) and (9), we get

$$\varepsilon_n^m k_1 = \varepsilon_n^{(1+1/m)} \frac{k}{(k_1)^{1/m}}$$

or $$\varepsilon_n^m = \varepsilon_n^{(1+1/m)} \frac{k}{(k_1)^{(1+1/m)}}. \quad ...(10)$$

Now choose k and k_1 such that

$$k = (k_1)^{(1+1/m)}.$$

Then from (10), we get

$$m = 1 + \frac{1}{m}$$

or $$m^2 - m - 1 = 0$$

or $$m = \frac{1 \pm \sqrt{5}}{2}.$$

Taking positive sign (as $m > 0$), then

$$m = \frac{1+\sqrt{5}}{2} = \frac{3{\cdot}236}{2} = 1{\cdot}618 = 1{\cdot}62.$$

Hence the order of convergence of secant method is **1·62**.

SOLVED EXAMPLES

Example 1. *Determine the root of the equation $f(x) = \cos x - xe^x = 0$ using the secant method upto four decimal places.*

Solution. Since $f(x) = \cos x - xe^x = 0$. ...(1)

Taking the initial approximations as $x_0 = 0, x_1 = 1$

Then $$f(x_0) = f(0) = \cos 0 - 0\,(e^0) = 1$$

and $$f(x_1) = f(1) = \cos 1 - 1\,(e) = -2{\cdot}1780.$$

By secant method, we have

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$= 1 - \frac{1-0}{-2.1780-1}(-2.1780)$$

$$= 1 - \frac{2.1780}{3.1780} = 1 - 0.6853.$$

$$x_2 = 0.3167.$$

$\therefore$

Now, $$f(x_2) = f(0.3147) = \cos(0.3147) - (0.3147)\, e^{0.3147}$$ [From (1)]

$$= 0.9509 - 0.4311 = 0.5198.$$

The second approximation to the root is given by

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

$$= 0.3147 - \frac{0.3147 - 1}{0.5198 + 2.1780}(0.5198)$$

$$= 0.3147 + \frac{0.3562}{2.6978} = 0.4467.$$

Now, $$f(x_3) = f(0.4467)$$

$$= \cos(0.4467) - (0.4467)\, e^{0.4467}$$ [From (1)]

$$= 0.9019 - 0.6983 = 0.2036.$$

The third approximation to the root is given by

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3)$$

$$= 0.4467 - \frac{0.4467 - 0.3147}{0.2036 - 0.5198}(0.2036)$$

$$= 0.4467 + \frac{0.0269}{0.3167}$$

$$x_4 = 0.5318.$$

Now, $$f(x_4) = f(0.5318)$$

$$= \cos(0.5318) - (0.5318)\, e^{0.5318}$$ [From (1)]

$$= 0.8619 - 0.9051 = -0.0432.$$

The fourth approximation to the root is given by

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} \cdot f(x_4)$$

$$= 0.5318 - \frac{0.5318 - 0.4467}{-0.0432 - 0.2036} \times (-0.0432)$$

$$= 0.5318 - \frac{0.0037}{0.2468}$$

$$= 0.5318 - 0.0150 = 0.5168.$$

Now, $$f(x_5) = f(0.5168)$$

$$= \cos(0.5168) - (0.5168)\, e^{0.5168}$$ [From (1)]

$$= 0.8694 - 0.8665 = 0.0029.$$

The fifth approximation to the root is given by

$$x_6 = x_5 - \frac{x_5 - x_4}{f(x_5) - f(x_4)} \cdot f(x_5)$$

$$= 0.5168 - \frac{0.5168 - 0.5318}{0.0029 + 0.0432} \times 0.0029$$

$$= 0.5168 + \frac{0.0000435}{0.0461} = 0.5177.$$

Now, $$f(x_6) = f(0.5177)$$

$$= \cos(0.5177) - (0.5177)\, e^{0.5177} \qquad \text{[From (1)]}$$

$$= 0.8690 - 0.8688 = 0.0002.$$

The sixth approximation to the root is given by

$$x_7 = x_6 - \frac{x_6 - x_5}{f(x_6) - f(x_5)} \cdot f(x_6)$$

$$= 0.5177 - \frac{0.5177 - 0.5168}{0.0002 - 0.0029} \times 0.0002$$

$$= 0.5177 + 0.00006 = 0.51776.$$

Hence, the root of the equation $\cos x - xe^x = 0$ correct to four decimal places is 0.5177.

Example 2. *Find a root of the equation $x - e^{-x} = 0$ correct to three decimal places by the secant method.*

Solution. Let $$f(x) = x - e^{-x} = 0 \qquad \text{...(1)}$$

Then $$f(0) = -1$$

and $$f(1) = 1 - e^{-1} = 0.6321$$

Taking $$x_0 = 0,\ x_1 = 1.$$

Applying secant method, the first approximation to the root is given by

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \times f(x_1)$$

$$= 1 - \frac{1 - 0}{0.6321 + 1} \times 0.6321$$

$$= 1 - \frac{0.6321}{1.6321} = 0.6127.$$

Now, $$f(x_2) = f(0.6127)$$

$$= (0.6127) - e^{-0.6127} \qquad \text{[From (1)]}$$

$$= 0.6127 - 0.5419 = 0.0708.$$

The second approximation to the root is given by

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} \cdot f(x_2)$$

$$= 0.6127 - \frac{0.6127 - 1}{0.0708 - 0.6321} \times 0.0708$$

$$= 0.6127 - \frac{0.0274}{0.5613}$$

$$= 0.6127 - 0.0488 = 0.5639.$$

Now, $$f(x_3) = f(0.5639) = 0.5639 - e^{-0.5639} \qquad \text{[From (1)]}$$

$$= 0.5639 - 0.5690 = -0.0051.$$

The third approximation to the root is given by

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} \cdot f(x_3)$$

$$= 0.5639 - \frac{0.5639 - 0.6127}{-0.0051 - 0.0708} \times (-0.0051)$$

$$= 0.5639 + \frac{0.00025}{0.0759}$$

$$= 0.5639 + 0.0033 = 0.5672.$$

Now, $f(x_4) = f(0.5672) = 0.5672 - e^{-0.5672}$ [From (1)]

$$= 0.5672 - 0.5671 = 0.0001.$$

The fourth approximation to the root is given by

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} \cdot f(x_4)$$

$$= 0.5672 - \frac{0.5672 - 0.5639}{0.0001 + 0.0051} \times 0.0001$$

$$= 0.5672 - 0.00063 = 0.5670.$$

Hence, the approximate root correct to three decimal places is 0.567.

EXERCISE 3

1. Find a real root of the following equations by secant method :
   (i) $x^3 + x^2 + x + 7 = 0$ (ii) $e^x - x = 0$
   (iii) $x^6 - x^4 - x^3 - 1 = 0$.
2. Using Regula-falsi method, compute the real root of the following equations :
   (i) $x^3 - 4x - 9 = 0$ (ii) $x^6 - x^4 - x^3 - 1 = 0$
   (iii) $xe^x = 2$ (iv) $\cos x = 3x - 1$
   (v) $x^3 - x - 4 = 0$.
3. Find a real root of the equation $3x + \sin x - e^x = 0$ by the method of false position correct to four decimal palces.
4. Find the root of the equation $xe^x = \cos x$ in the interval (0, 1) using Regula-falsi method correct to four decimal places.
5. Find the root of the equation $\tan x + \tanh x = 0$ in the interval (1.6, 3.0) correct to three decimal places by Regula-falsi method.
6. Find a real root of the equation $x \log_{10} x = 1.2$ by Regula-falsi method correct to four decimal places.
7. Apply False-position method to find smallest positive root of the equation $x - e^{-x} = 0$ correct to three decimal places.
8. Solve the following equations by Regula-falsi method :
   (i) $x^3 + x - 1 = 0$ near $x = 1$ (ii) $2x - \log_{10} x = 7$ in (3.5, 4)
   (iii) $x^3 + x^2 - 3x - 3 = 0$ in (1, 2) (iv) $x^3 - 3x + 4 = 0$ in (– 2, – 3)
   (v) $x^3 - 5x - 7 = 0$ (vi) $x^3 - x^2 - 2 = 0$
   (vii) $x = \tan x$ (viii) $x^6 - x^4 - x^3 - 3 = 0$
   (ix) $(5 - x)\, e^x = 5$ near $x = 5$ (x) $x^4 + x^3 - 7x^2 - x + 5 = 0$ in (2, 3)

(xi) $x^4 - x^3 - 2x^2 - 6x - 4 = 0$ (xii) $x^3 + x - 3 = 0$

(xiii) $x^3 - 5x + 3 = 0$ (xiv) $x^3 - x - 4 = 0$.

9. Using False-position method, find x when $x^2 - 9 = 0$.

10. Find real cube root of 18 by Regula-falsi method.

ANSWERS

1. (i) − 2·0625 (ii) 0·6190 (iii) 1·404

2. (i) 2·7065 (ii) 1·404 (iii) 0·853 (iv) 0·6071 (v) 1·7963

3. 0·3604 **4.** 0·5177 **5.** 2·365 **6.** 2·7406 **7.** 0·567

8. (i) 0·6823 (ii) 3·7892 (iii) 1·7321 (iv) – 2·1958 (v) 2·7473

(vi) 1·6956 (vii) 4·4934 (viii) 1·5018 (ix) 4·96511 (x) 2·06085

(xi) 2·73205 (xii) 1·2134 (xiii) 0·6566 (xiv) 1·7963

9. 3 **10.** 2·62074.